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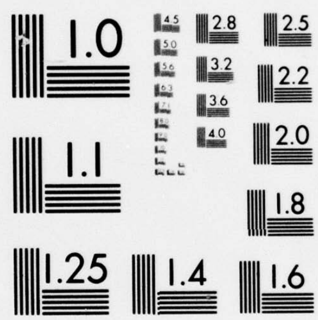

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UNIQUENESS OF SOLUTIONS TO $-\Delta u - qu = 0$

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ABSTRACT

We give conditions for the function $q(x)$ which guarantee uniqueness of solutions to $-\Delta u(x) + \lambda_0 u(x) - q(x)u(x) = f$, where $\lambda_0 \geq 0$ is a constant. We consider separately the cases for solutions in H^1 and L^2. In fact we even prove existence of H^1 solutions under the additional assumptions $q(x) \geq 0$ and $f \in L^2$. We apply our results to the problem of essential selfadjointness for a class of Schrödinger operators and we are able to prove an interesting generalization of a result due to Kato.

Gaveau has proved a result similar to ours on L^2 uniqueness. His proof, however, depends on a probabilistic argument, while in contrast we do not use any probability theory in our proofs.

AMS (MOS) Subject Classifications: 35A05, 35J10

Key Words: Bessel potential, Distribution, Measure, Schrödinger operator

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

The Schrödinger wave equation is of fundamental importance in quantum mechanics. In the early 1930's von Neumann made use of the theory of self-adjoint operators on a Hilbert space to place quantum mechanics on a firm mathematical foundation. Thus it is important to know under what conditions a Schrödinger operator with singular potential is selfadjoint on a Hilbert space. The obvious Hilbert space is L^2, the space of square-integrable functions.

We consider Schrödinger operators of the form

$$T \equiv \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - ib_j(x) \right)^2 ,$$

with a singular potential $q(x)$. The operator $T + q$ is difficult to interpret as an operator on L^2 since L^2 functions are not necessarily differentiable. This difficulty is circumvented by the following device: define

$$(\overline{T + q})(u) \equiv w \quad \text{for} \quad u,w \in L^2$$

if there is a sequence of infinitely differentiable functions $\phi_i \in C_0^\infty$ such that $\phi_i \to u$ in L^2 and $(T + q)(\phi_i) \to w$ in L^2. The problem now is that $\overline{T + q}$ may not be well defined. If it is then we say $T + q$ is <u>essentially</u> <u>selfadjoint</u> and $\overline{T + q}$ is a selfadjoint operator on the Hilbert space L^2. Our paper generalizes results of T. Kato on conditions which guarantee $T + q$ is essentially selfadjoint.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

UNIQUENESS OF SOLUTIONS TO $-\Delta u - qu = 0$

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We give conditions on $q(x)$ which guarantee that if $u(x) \in L^2(\mathbb{R}^n)$ and

$$(0.1) \quad -\Delta u(x) + \lambda_0 u(x) - q(x)u(x) = 0,\ \lambda_0 \geq 0 \text{ a constant },$$

then $u(x) \equiv 0$. In fact, we show that if there exists a constant θ_0 such that

$$(0.2) \quad (-\Delta + \lambda I)^{-1}(q^+)(x) \leq \theta_0 < 1, \text{ for all } \lambda > \lambda_0 ,$$

then $u(x) \in L^2(\mathbb{R}^n)$ satisfies (0.1) if and only if $u(x) \equiv 0$. This result is contained in our Theorem 2.2. (Gaveau has proved this result by probabilistic means. Our techniques are entirely different from his and use no probability theory.)

Our results can also be applied to a larger class of second order elliptic operators. Indeed, remark 2 of section 1 gives a class of operators to which our results can be extended. Furthermore, by using a beautiful lemma due to T. Kato (the lemma and proof are found in [2]) it is possible to apply uniqueness results about $-\Delta + \lambda_0 - q$ to a larger class of Schrödinger operators. A statement of Kato's lemma is found in section 2 of this paper.

The paper proceeds as follows. In section 1 we obtain existence and uniqueness results for

$$(0.3) \quad -\Delta u + \lambda_0 u - \mu u = f$$

where μ is a non-negative measure. We obtain these results for $f \in L^2$ and $u \in H^1$. Section 2 contains the L^2 uniqueness result for (0.2). The proof depends heavily on the material developed in section 1. Finally, section 3 contains an application of the results of section 2. We prove a strong generalization of the results found in [2] on essential self-adjointness.

1. Existence and uniqueness of solutions in H^1

We use the following notation throughout this paper.

$$C_0^\infty \equiv C_0^\infty(\mathbb{R}^n),\ C^\infty \equiv C^\infty(\mathbb{R}^n),$$

$$L^p \equiv L^p(\mathbb{R}^n),\ L^p_{loc} \equiv L^p_{loc}(\mathbb{R}^n),$$

$$H^s \equiv H^s(\mathbb{R}^n),\ H^s_{loc} \equiv H^s_{loc}(\mathbb{R}^n),$$

$$L^p(\mu) \equiv L^p(\mu;\mathbb{R}^n) \quad \text{for } \mu \text{ a measure on } \mathbb{R}^n,$$

$$(\psi_1,\psi_2) \equiv \int_{\mathbb{R}^n} \psi_1 \cdot \bar{\psi}_2(x)dx \quad \text{for } \psi_i \in L^2,\ i = 1,2 ,$$

$$(\psi_1,\psi_2)_\mu \equiv \int_{\mathbb{R}^n} \psi_1 \cdot \bar{\psi}_2(x)d\mu(x) \quad \text{for } \psi_i \in L^2(\mu),\ i = 1,2 .$$

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Until further notice (near the end of section 2) all functions and measures are real valued. We shall also use the Bessel potentials,

$$J_\lambda \equiv F^{-1}((4\pi^2|\xi|^2 + \lambda)), \quad \lambda > 0 ,$$

where F^{-1} is the inverse Fourier transform. For each measurable function $\phi : \mathbb{R}^n \to \mathbb{R}$ define

$$(-\Delta + \lambda I)^{-1}\phi(x) \equiv \int_{\mathbb{R}} J_\lambda(x - y)\phi(y)dy ,$$

whenever $J_\lambda(x - \cdot)\varphi(\cdot) \in L^1$ for a.e. $x \in \mathbb{R}^n$. The functions J_λ have the property,

(1.1) $\qquad$ if $\lambda > \lambda'$ then $0 \le J_\lambda \le J_{\lambda'}$.

(More information about J_λ can be found in [3].)

The following assumption will be used.

(1.2) $\begin{cases} \mu \text{ is a non-negative measure on } \mathbb{R}^n; \text{ there is} \\ \text{a constant } \lambda_0 \ge 0 \text{ such that for all } \lambda > \lambda_0 \\ (-\Delta + \lambda I)^{-1}(\mu)(x) \le \theta_0 \le 1 \text{ for some constant } \theta_0 \ge 0 \end{cases}$

and whenever λ_0 or θ_0 appear in this paper they are the constants in (1.2). (We define

$(-\Delta + \lambda I)^{-1}(\mu)(x) \equiv \int_{\mathbb{R}^n} J_\lambda(x - y)d\mu(y)$.)

Although we shall delay the proof until later we now state the main theorem of this section:

Theorem 1.1. Let (1.2) hold and let $\phi \in H^1$. Then $\phi \in L^2(\mu)$ and

(1.3) $\qquad (\nabla\phi,\nabla\phi) + \lambda_0(\phi,\phi) - (\phi,\phi)_\mu \ge (1 - \theta_0)(\nabla\phi,\nabla\phi)$,

where $\nabla\phi \equiv \left(\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n}\right)$.

Corollary 1.2. Let (1.2) hold, let $\lambda > \lambda_0$, $f \in L^2$ and $\theta_0 < 1$. Then there is a unique function $w \in H^1$ such that

(1.4) $\qquad (-\Delta w + \lambda w,\phi) - (w,\phi)_\mu = (f,\phi)$,

for all $\phi \in C_0^\infty$. Furthermore, there is a constant $C(\theta_0,\lambda)$ such that

(1.5) $\qquad \|w\|_{H^1} \le C(\theta_0,\lambda)\|f\|_{L^2}$.

Proof. Using Theorem 1.1 we see that for any $\phi \in H^1$

(1.6) $\quad (\nabla\phi,\nabla\phi) + \lambda(\phi,\phi) - (\phi,\phi)_\mu \ge (1-\theta_0)(\nabla\phi,\phi) + (\lambda-\lambda_0)(\phi,\phi)$.

The existence and uniqueness of w and the estimate (1.5) all follow from the Riesz theorem in the usual way.

The following remarks (verifications of which are of varying degrees of difficulty) represent several directions in which generalizations of our results can be made.

Remark 1. Suppose we replace $\mathbb{R}^n$ by a domain Ω and change (1.2) to

(1.7) $\begin{cases} \mu \text{ is a non-negative measure on } \Omega, \text{ there are} \\ \text{constants } \lambda_0 \ge 0,\ 1 \ge \theta_0 \ge 0 \text{ and a measurable} \\ \text{function } v(x),\ 0 \le v(x) \le \theta_0 \text{ such that} \\ -\Delta v + \lambda_0 v = \mu \text{ (in the sense of distributions).} \end{cases}$

Then all our assertions remain valid with H^1 replaced by $H_0^1(\Omega)$ and L^2 replaced by $L^2(\Omega)$. Although it may not be apparent, if $\Omega = \mathbb{R}^n$ then (1.2) and (1.7) are equivalent.

Remark 2. Let A be the partial differential operator

$$A\phi(x) \equiv \sum_{i,j=1}^{n} -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial \phi}{\partial x_j}(x)\right) \quad \text{for } \phi \in C_0^\infty, \tag{1.8}$$

where $(a_{ij}(x))$ is uniformly positive definite and symmetric and the $a_{ij}(x)$ are real valued, smooth and bounded. If we now replace $-\Delta$ by A and (1.2) by (1.7) then analogous results are valid.

We return again to the business at hand, namely, the proof of Theorem 1.1. We remind the reader of a definition:

$$\left\{\begin{array}{l} \{b_j(x)\}_{j=1}^\infty \subset L^1_{loc} \text{ converges in } D'(\mathbb{R}^n) \text{ to a} \\ \text{measure } \mu \text{ if for any } \phi \in C_0^\infty,\ (\phi, b_j) \to (\phi, 1)_\mu \\ \text{as } j \to \infty . \end{array}\right. \tag{1.9}$$

Lemma 1.3. Let (1.2) hold and let λ be a constant. Then:

(i) There exists a sequence of functions $\{a_j\}_{j=1}^\infty \subset C_0^\infty \cap L^\infty$ such that $a_j(x) \geq 0$ $\forall j$, $\{a_j\}_{j=1}^\infty$ converges in $D'(\mathbb{R}^n)$ to μ and if $\lambda > \lambda_0$ then

$$(-\Delta + \lambda I)^{-1}(a_j)(x) \leq \theta_j < \theta_0 . \tag{1.10}$$

(ii) Let $\tilde{a}(x) \in L^\infty$, $\tilde{a}(x) \geq 0$ and $(-\Delta + \lambda I)^{-1}(\tilde{a})(x) \leq \tilde{\theta} < 1$. Then for any function $g \in L^2$, $g \geq 0$ which satisfies

$$(-\Delta + \lambda I)^{-1}(g)(x) \in L^\infty , \tag{1.11}$$

there is a function $G \in H^2_{loc} \cap L^\infty$ such that

$$\left\{\begin{array}{l} -\Delta G + \lambda G - \tilde{a} G = g \\ 0 \leq G(x) \leq (1 - \tilde{\theta})^{-1} \|(-\Delta + \lambda I)^{-1}(g)\|_{L^\infty} . \end{array}\right. \tag{1.12}$$

Furthermore, if in addition $\tilde{a}(x), g(x) \in C^\infty$ then $G \in C^\infty$.

Proof of (i). Simply define $a_j(x)$ by

$$a_j(x) \equiv \left\{\left(\frac{j-1}{j}\right) \int_{\mathbb{R}^n} K_j e^{-j|x-y|^2} d\mu(y)\right\} \xi_j(x)$$

where K_j is the normalizing constant

$$(K_j)^{-1} \equiv \int_{\mathbb{R}^n} e^{-j|x|^2} dx ,$$

and $\xi_j \in C_0^\infty$, $\xi_j \to 1$ uniformly on compact sets. It amounts to an exercise to verify that the sequence $\{a_j\}_{j=1}^\infty$ just defined has the required properties.

Proof of (ii). We shall obtain G as the limit of a sequence of functions $\{G_i\}_{i=1}^\infty$. We define G_i inductively by

$$\left\{\begin{array}{l} G_0(x) \equiv 0 , \\ G_{i+1}(x) \equiv (-\Delta + \lambda I)^{-1}(\tilde{a} G_i + g)(x) . \end{array}\right. \tag{1.13}$$

First we establish that G_i is monotone increasing. Indeed, setting

$$\bar{G}_i = G_i - G_{i-1}$$

we see $\bar{G}_i$ satisfies the equation

$$\bar{G}_{i+1} = (-\Delta + \lambda I)^{-1}(\tilde{a} \bar{G}_i) . \tag{1.14}$$

Since $\bar{G}_1 = G_1 - G_0 = G_1 \geq 0$, we conclude by induction that

(1.15) $\qquad \bar{G}_i \geq 0$ for all i,

which proves the monotonicity.

Next, let us estimate $\|\bar{G}_i\|_{L^\infty}$. From (1.14) we see that since $\tilde{a}(x) \geq 0$

$$\|\bar{G}_{i+1}\|_{L^\infty} \leq \|(-\Delta + \lambda I)^{-1}(\|\bar{G}_i\|_{L^\infty}\tilde{a})\|_{L^\infty}$$

$$\leq \|\bar{G}_i\|_{L^\infty} \|(-\Delta + \lambda I)^{-1}(\tilde{a})\|_{L^\infty} .$$

By the above and the hypothesis on $\tilde{a}(x)$ we have the inequality

$$\|\bar{G}_{i+1}\|_{L^\infty} \leq \|\bar{G}_i\|_{L^\infty}\tilde{\theta} .$$

Using induction we conclude

(1.16) $\qquad \|\bar{G}_{i+1}\|_{L^\infty} \leq \|(-\Delta + \lambda I)^{-1}(g)\|_{L^\infty}(\tilde{\theta})^i .$

From the inequality above we see that $\{G_i\}_{i=1}^\infty$ is a Cauchy sequence in the L^∞ norm. Therefore the sequence has a limit which we denote by $G(x)$. By (1.15) and (1.16) we have

$$0 \leq G(x) \leq (1 - \tilde{\theta})^{-1} \|(-\Delta + \lambda I)^{-1}(g)\|_{L^\infty} .$$

So it remains only to show $G \in H^2_{loc}$. This fact can be proved by noting that for all i, $G_i \in H^2_{loc}$ since

$$-\Delta G_i + \lambda G_i = \tilde{a}G_{i-1} + g$$

and because $\tilde{a}(x), g(x) \in L^\infty$. Therefore

(1.17) $\quad -\Delta G + \lambda G - \tilde{a}G = g$ (in the sense of distributions) .

Since $\tilde{a}(x)$, $g(x)$, $G(x) \in L^\infty$ it follows from a result of

Friedrichs' that (1.17) implies

$$G \in H^2_{loc} .$$

All that now remains is to consider the case $\tilde{a}(x)$, $g(x) \in C^\infty$ also. In this case, however, it follows by induction that

$$G \in H^{2\ell}_{loc} \text{ for all positive integers } \ell .$$

This completes the proof of the lemma.

We end this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. It will be sufficient to prove that for every $\phi \in C_0^\infty$

(1.18) $\quad (\nabla\phi,\nabla\phi) + \lambda_0(\phi,\phi) - (\phi,\phi)_\mu \geq (1 - \theta_0)(\nabla\phi,\nabla\phi)$.

We shall prove (1.18) by constructing special functions $W_j(x) \in C^\infty$ and decomposing ϕ as

(1.19) $\qquad \phi = W_j\psi_j$,

and then expanding $(\nabla(W_j\psi_j),\nabla(W_j\psi_j))$ appropriately (using the properties of W_j).

We define $W_j(x)$ as follows. Let $\{a_j\}_{j=1}^\infty$ be the sequence of functions whose existence is guaranteed by (i) of Lemma 1.3. We let G_j be the solution of

(1.20) $\quad -\Delta G_j + \lambda G_j - a_jG_j = a_jf_\lambda$ for any $\lambda > \lambda_0$,

where $f_\lambda(x) = \frac{1}{2n}\left(\sum_{k=1}^{n} (e^{\sqrt{\lambda}x_k} + e^{-\sqrt{\lambda}x_k})\right)$,

which we can do by (ii) of Lemma 1.3 and where $\tilde{a} = a_j$ and

$g = a_j f_\lambda$. Finally, we define W_j as

$$W_j \equiv G_j + f_\lambda .$$

Notice that by (1.20) and (ii) of Lemma 1.3

$$(1.21) \qquad \begin{cases} W_j \in L^\infty \cap C^\infty, \quad W_j(x) \geq 1 , \\ -\Delta W_j + \lambda W_j - a_j W_j = 0 . \end{cases}$$

We now evaluate $(\nabla\phi,\nabla\phi)$ defining $\psi_j \in C_0^\infty$ by (1.19).

We have using integration by parts,

$$\begin{aligned}(\nabla\phi,\nabla\phi) &= (\nabla(W_j\psi_j),\nabla(W_j\psi_j)) , \\ &= (W_j\psi_j,-\Delta(W_j\psi_j)) , \\ &= (W_j\psi_j,-W_j(\Delta\psi_j) - 2(\nabla W_j)(\nabla\psi_j)) \\ &\qquad + (W_j\psi_j,(-\Delta W_j)\psi_j) , \\ &= (\psi_j,-W_j^2(\Delta\psi_j) - \nabla(W_j^2)\nabla\psi_j) + (W_j\psi_j,(-\Delta W_j)\psi_j) .\end{aligned}$$

Using integration by parts again and (1.21) we get

$$\begin{aligned}(\nabla\phi,\nabla\phi) + \lambda(\phi,\phi) &= (W_j(\nabla\psi_j),W_j(\nabla\psi_j)) + (W_j\psi_j,a_jW_j\psi_j) , \\ &\geq (\phi,a_j\phi) , \\ &\geq (\phi^2,a_j) .\end{aligned}$$

By (i) of Lemma 1.3 $\{a_j\}_{j=1}^\infty$ converges in $\mathcal{D}'(\mathbb{R}^n)$ to μ and so

$$(\nabla\phi,\nabla\phi) + \lambda(\phi,\phi) \geq (\phi^2,1)_\mu = (\phi,\phi)_\mu .$$

Next by noting that the above inequality holds for all $\lambda > \lambda_0$ we see

$$(1.22) \qquad (\nabla\phi,\nabla\phi) + \lambda_0(\phi,\phi) \geq (\phi,\phi)_\mu .$$

Now if (1.2) holds for μ with $\theta_0 < 1$ then (1.2) holds for $\tilde{\mu} = \theta_0^{-1}\mu$ with $\tilde{\theta}_0 = 1$. Thus (1.22) holds for μ replaced by $\tilde{\mu}$ and this gives us

$$(1.23) \qquad (\nabla\phi,\nabla\phi) + \lambda_0(\phi,\phi) \geq (\phi,\phi)_{\tilde{\mu}} .$$

Using the definition of $\tilde{\mu}$ and rearranging (1.23) we get

$$(\nabla\phi,\nabla\phi) + \lambda_0(\phi,\phi) - \theta_0^{-1}(\phi,\phi)_\mu \geq 0 .$$

From this we have

$$\theta_0(\nabla\phi,\nabla\phi) + \lambda_0\theta_0(\phi,\phi) - (\phi,\phi)_\mu \geq 0$$

and by adding $(1 - \theta_0)(\nabla\phi,\nabla\phi)$ to both sides of the above inequality we get

$$(\nabla\phi,\nabla\phi) + \lambda_0\theta_0(\phi,\phi) - (\phi,\phi)_\mu \geq (1 - \theta_0)(\nabla\phi,\nabla\phi) .$$

From this (1.3) follows easily and this completes the proof of Theorem 1.1.

2. Uniqueness of solutions in L^2

From our previous work we shall prove two uniqueness results for functions in L^2 which satisfy an appropriate partial differential equation or inequality. A version of our second result has been proved in a recent paper by Gaveau.

We let $q(x) \in L^1_{loc}$, $q(x) \geq 0$. To such a function we associate a non-negative measure μ_q by setting

$$\mu_q(G) \equiv \int_G q(x)dx .$$

When we say (1.2) holds for q we mean (1.2) holds for μ_q.

For $q(x) \in L^1_{loc}$, $q(x) \geq 0$ we also define $q_j(x)$ by

$$(2.1) \qquad q_j(x) \begin{cases} q(x) & \text{if } q(x) < j \\ j & \text{if } q(x) \geq j \end{cases}.$$

Theorem 2.1. Let $q(x) \in L^1_{loc}$, $q(x) \geq 0$ and suppose (1.2) holds for q with $\theta_0 < 1$. If $u(x) \in L^2$, $u(x) \geq 0$, $qu(x) \in L^1_{loc}$, $\lambda > \lambda_0$ and

$$(2.2) \qquad -\Delta u + \lambda u - qu \leq 0 \quad \text{(in the sense of distributions)} ,$$

then $u(x) \equiv 0$.

Proof*. The idea of the proof is to solve a sequence of approximate adjoint equations to (2.2) for functions in some appropriate class and then show $u \equiv 0$ by using these with inequality (2.2). Let $g \in C^\infty_{0+} \equiv \{\psi \in C^\infty_0 | \psi \geq 0\}$; for each positive integer j let G^j be the solution of

$$(2.3) \qquad -\Delta G^j + \lambda G^j - q_j G^j = g ,$$

(where the $q_j(x)$ are defined by (2.1)) which we know we can solve uniquely by Corollary 1.2. By Corollary 1.2 and Lemma 1.3

$$(2.4) \qquad \begin{cases} \|G^j\|_{H^1} \leq C(\theta_0,\lambda)\|g\|_{L^2} , \\ 0 \leq G^j(x) \leq K, \text{ a constant independent of } j . \end{cases}$$

We note also that since $q_j \in L^\infty$

$$(2.5) \qquad G^j \in H^2 ,$$

however, the norm of G^j in H^2 may explode as $j \to \infty$.

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*We give a modification due to H. Brezis of our original proof.

We introduce the functions $\zeta_R(x) \in C^\infty_0$

$$(2.6) \qquad \begin{cases} 0 \leq \zeta_R(x) \leq 1 , \\ \zeta_R(x) = 0 \text{ if } |x| > R + 1 , \\ \zeta_R(x) = 1 \text{ if } |x| < R , \end{cases}$$

and $\zeta_R(x)$ has first and second derivatives uniformly bounded in R for all $R > 0$.

Thus by (2.3)

$$(g,u) = (u,-\zeta_R \Delta G^j + \zeta_R \lambda G^j - \zeta_R q_j G^j) ,$$

for R sufficiently large. From this we get

$$(2.7) \qquad (g,u) = (u,-\Delta(\zeta_R G^j) + \lambda(\zeta_R G^j) - q(\zeta_R G^j)) + (u,(\Delta\zeta_R)G^j + 2(\nabla\zeta_R)\nabla G^j) + (u,(q - q_j)\zeta_R G^j) .$$

Now, $\zeta_R G^j(x) \in H^2 \cap L^\infty$, $\zeta_R G^j(x) \geq 0$ and $\zeta_R G^j(x)$ has compact support. Choosing

$$\psi_\ell \to \zeta_R G^j \text{ in } H^2 \cap L^\infty ,$$

and by integrating (2.2) multiplied by ψ_ℓ we conclude upon letting $\ell \to \infty$ that,

$$(u,-\Delta(\zeta_R G^j) + \lambda(\zeta_R G^j) - q(\zeta_R G^j)) \leq 0 .$$

From this and (2.7) we get

$$(u,g) \leq (u,(\Delta\zeta_R)G^j + 2(\nabla\zeta_R)\nabla G^j) + (u,(q - q_j)\zeta_R G^j) .$$

For R fixed we see that

$$(2.8) \qquad u(q - q_j)\zeta_R G^j \to 0 \text{ a.e. as } j \to \infty .$$

By (2.1) and (2.4) we have

$$|u(q - q_j)\zeta_R G^j| \le K|uq\zeta_R| .$$

From this, (2.8), since $uq \in L^1_{loc}$, and because $\zeta_R \in C_0^\infty$ we may apply the Lebesgue dominated convergence theorem to conclude

$$(u, (q - q_j)\zeta_R G^j) \to 0 \quad \text{as} \quad j \to \infty .$$

Using this in (2.7) gives us

$$(u,g) \le (u, (\Delta\zeta_R)G + 2(\nabla\zeta_R)\nabla G) . \tag{2.9}$$

By hypothesis $\Delta\zeta_R$, $\nabla\zeta_R$ are uniformly bounded independent of R and so there is a positive constant C' such that C' is independent of R and

$$\max\{\|\Delta\zeta_R\|_{L^\infty}, \|\nabla\zeta_R\|_{L^\infty}\} \le C' . \tag{2.10}$$

We see from (2.6) that

$$\operatorname{supp}(|\nabla\zeta_R| + |\Delta\zeta_R|) \subset A_R \equiv \{x \in \mathbb{R}^n | R < |x| < R + 1\} .$$

Using this fact, (2.4) and (2.10) we derive

$$(u,g) \le C'C(\theta_0,\lambda)\|g\|_{L^2} \int_{A_R} |u|^2(x)dx , \tag{2.11}$$

from (2.9). By assumption $u \in L^2$ and therefore

$$\int_{A_R} |u|^2(x)dx \to 0 \quad \text{as} \quad R \to \infty .$$

Using this in (2.11) we get

$$(u,g) \le 0 .$$

Since the above inequality is true for all $g \in C_{0+}^\infty$ we see that

$$u(x) \equiv 0 .$$

This completes the proof of Theorem 2.1.

In order to prove our next theorem we require the use of a lemma due to Kato. With the aid of this lemma we can reduce the proof of Theorem 2.2 to the case just handled. For the rest of this paper functions and constants will be complex valued unless specifically stated otherwise.

[2]; Lemma A, pg. 138. Let

$$L = \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_j} - ib_j(x)\right) a_{jk}(x) \left(\frac{\partial}{\partial x_k} - ib_k(x)\right) , \tag{2.12}$$

where a_{jk} and b_j are real valued functions in $C^1(\Omega)$, Ω being an open set in $\mathbb{R}^n$. Assume that $(a_{jk}(x))$ is a positive definite symmetric matrix for each $x \in \Omega$. If u and Lu are in $L^1_{loc}(\Omega)$ then

$$L_0|u| \ge \operatorname{Re}[(\operatorname{sign} \bar{u})Lu] \tag{2.13}$$

where L_0 is the operator L with $b_j = 0$, $j = 1,\dots,n$. (By definition $\operatorname{sign}(\bar{u}(x)) = \bar{u}(x)/|u(x)|$ if $u(x) \ne 0$ and $\operatorname{sign}(\bar{u}(x)) = 0$ if $u(x) = 0$.) Note that we are now allowing $u(x)$ to be a complex valued function.

The following result is now easily proved.

<u>Theorem 2.2</u>. Let $q_1(x) \ge 0$ and $q_2(x) \ge 0$ be real valued and measurable on $\mathbb{R}^n$ and suppose (1.2) holds for $q_2(x)$. If $u(x) \in L^2$, $(q_1 + q_2)u \in L^1_{loc}$, $\operatorname{Re}(\zeta) > \lambda_0$, $\theta_0 < 1$ and

$$\left(-\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - ib_j(x)\right)^2 + \zeta + q_1(x) - q_2(x)\right)u = 0 \tag{2.14}$$

(in the sense of distributions),

(where $b_j(x)$, $j = 1,\dots,n$ are real valued continuously

differentiable) then

$$u(x) \equiv 0 .$$

Proof**. By Lemma A (from [2]) with $\Omega = \mathbb{R}^n$ and $a_{jk}(x) = \delta_{jk}$ and by (2.14) we conclude

$$\mathrm{Re}(-\Delta|u| + \zeta|u| + q_1|u| - q_2|u|) \leq 0$$

(in the sense of distributions) .

This easily yields

(2.15) $$-\Delta|u| + \mathrm{Re}(\zeta)|u| - q_2|u| \leq 0$$

(in the sense of distributions) .

Now we merely apply Theorem 2.1 to $|u|(x)$ with $q(x) = q_2(x)$ and $\lambda = \mathrm{Re}(\zeta)$. We conclude

$$|u|(x) \equiv 0 ,$$

and this completes the proof of the theorem.

3. Application to Schrödinger operators

Consider the Schrödinger operator

(3.1) $$T \equiv \sum_{j=1}^{n} -\left(\frac{\partial}{\partial x_j} - ib_j(x)\right)^2 + q_1(x) - q_2(x)$$

where b_j and q_i are real valued functions on $\mathbb{R}^n$ such that

$$q_i \in L^2_{loc},\ b_j \in C^1, \quad i = 1,2;\ j = 1,\dots,n .$$

Denote by T_0 the operator T with $b_j = 0$, $j = 1,\dots,n$. We suppose T is defined on L^2 with values in D' (the space of distributions on $\mathbb{R}^n$).

**This argument was suggested by H. Brezis.

Let $\dot{T} = T|_{C_0^\infty}$ i.e. the restriction of T to $D = C_0^\infty$. It is clear that $\dot{T}$ is a symmetric operator and in [2] Kato has shown that under certain conditions on q_1 and q_2, $\dot{T}$ is essentially selfadjoint. The conditions we are interested in are

(3.2) $q_1(x) \geq -q^*(|x|)$, where $q^*(r) \geq 0$ is monotone nondecreasing in $r > 0$ and

$$q^*(r) = O(r^2) \quad \text{as} \quad r \to \infty .$$

(3.3) $q_2 \in L^2_{loc}$, with

$$\int_{|x|\leq r} |q_2(x)|^2 dx \leq K^2 r^{2s} \qquad 1 \leq r < \infty$$

where K and s are some constants and

(3.4) $$\int_{|y|\leq r} |q_2(x-y)|\,|y|^{2-n} dy \to 0 \quad \text{as} \quad r \to 0$$

uniformly in $x \in \mathbb{R}^n$,

where $|y|^{2-n}$ should be replaced by $|\log|y||$ if $n = 2$.

We shall show that (3.3) and (3.4) can be replaced by the simpler condition

(3.5) $q_2 \in L^2_{loc}$ and there exist constants $r_0 > 0$, $1 > \alpha_0 > 0$ such that

$$\frac{\Gamma(\frac{n}{2})}{2(n-2)\pi^{\frac{n}{2}}} \int_{|y|\leq r_0} |q_2(x-y)|\,|y|^{2-n} dy \leq \alpha_0 \quad \text{for all} \quad x \in \mathbb{R}^n ,$$

where $|y|^{2-n}$ should be replaced by $|\log|y||$ if $n = 2$ and the leading constant appropriately modified. This leads to the following theorem.

Theorem 3.1. Let $\dot{T}$ be the operator defined on D. If (3.2) holds for $q_1(x)$ and if (3.5) holds for $q_2(x)$ then $\dot{T}$ is essentially selfadjoint.

Remark 3. Kato notes that if $n \geq 5$ then (3.3) and (3.4) may be replaced by

$$q_2 \in L^{n/2} . \tag{3.6}$$

In a private communication Brezis has shown that for $n \leq 4$ there are conditions analogous to (3.6) which may be used to replace (3.3) and (3.4). For example, if $n = 3$ then we may replace (3.3) and (3.4) by

$$q_2 \in L^{3/2} \cap L^2_{loc} . \tag{3.7}$$

His proof depends on a modification of the argument used to prove Theorem 2.1 and the results in [2].

Proof of Theorem 3.1 for $q^* =$ constant. We shall give an outline of our method of attack. We show that if $Re(\zeta)$ is sufficiently large then $T + \zeta : L^2 \to D'$ is a one to one map. As in [2] this implies that $\dot{T}$ is essentially selfadjoint. In order to show $T + \zeta$ is one to one we shall prove that if (3.5) holds then there exists a $\lambda_0 \geq 0$ such that for any $\lambda > \lambda_0$

$$(-\Delta + \lambda I)^{-1}(|q_2|)(x) \leq \alpha_0 + \frac{1 - \alpha_0}{2} . \tag{3.8}$$

We may then apply Theorem 2.2 to $T + \zeta$ to achieve the desired result. We now verify the steps in the outline via a series of lemmas. (For convenience we assume $n \geq 3$ in what follows.)

Lemma 3.2. Assume (3.5) holds for $q_2(x)$. Then there exists a positive constant K such that

$$\int_{|y|<r} |q_2(x - y)| dy < Kr^n \quad \text{for all} \quad x \in \mathbb{R}^n, \quad r \geq r_0 . \tag{3.9}$$

Proof. Set $r_1 = \frac{r_0}{\sqrt{n}}$, then

$$\int_{|y_1|<r_1,\dots,|y_n|<r_1} |q(x - y)| dy \leq \int_{|y|<\sqrt{n}\, r_1} |q(x - y)| dy$$

$$\leq C_n^{-1}(\sqrt{n}\, r_1)^{n-2} \int_{|y|<\sqrt{n}\, r_1} C_n |q(x - y)| |y|^{-n+2} dy$$

where $C_n = \frac{\Gamma(\frac{n}{2})}{2(n - 2)\pi^{n/2}}$. By (3.5) and the definition of r_1 we have for some constants K_1 and K_2

$$\int_{|y_1|<r_1,\dots,|y_n|<r_1} |q(x - y)| dy \leq K_1 r_1^{n-2} \alpha_0 \leq K_2 r_1^n .$$

Using the fact that K_2 is independent of x we see

$$\int_{|y_1|<jr_1,\dots,|y_n|<jr_1} |q(x - y)| dy \leq K_2 j^n r_1^n = K_2 (jr_1)^n .$$

Finally, by increasing K_2 if necessary we conclude that there is a constant K such that

$$\int_{|y|<r} |q(x - y)| dy \leq Kr^n \quad \text{for all} \quad x \in \mathbb{R}^n, \quad r \geq r_0 .$$

This completes the proof of the lemma.

We remind the reader of the definition in section 1 of $J_\lambda(x)$. It is verified by change of variables that

$$J_{\lambda^2}(x) = \lambda^{n-2} J_1(\lambda x) . \tag{3.10}$$

In [3] it is shown that there exist positive constants δ and R such that

$$\text{(3.11)} \qquad \text{if } |x| > R \text{ then } J_1(x) \le e^{-\delta|x|} .$$

Lemma 3.3. Assume (3.5) holds for $q_2(x)$. Then there exists a constant $\lambda_0 > 0$ such that

$$\text{(3.12)} \qquad (-\Delta + \lambda_0 I)^{-1}(|q_2|)(x) \le \alpha_0 + \frac{1-\alpha_0}{2} .$$

Proof. We must show that for some $\lambda_0 > 0$

$$\text{(3.13)} \qquad \int_{\mathbb{R}^n} J_{\lambda_0}(y)|q_2(x-y)|dy \le \alpha_0 + \frac{1-\alpha_0}{2} .$$

Now

$$\int_{\mathbb{R}^n} J_{\lambda_0}(y)|q_2(x-y)|dy = \int_{|y|<r_0} J_{\lambda_0}(y)|q_2(x-y)|dy + \int_{|y|\ge r_0} J_{\lambda_0}(y)|q_2(x-y)|dy .$$

Using (3.5), the equality above, and since $0 \le J_{\lambda_0}(y) \le C_n|y|^{-n+2}$ we conclude that it is sufficient to prove

$$\text{(3.14)} \qquad \int_{|y|\ge r_0} J_{\lambda_0}(y)|q_2(x-y)|dy \le \frac{1-\alpha_0}{2}, \quad \text{for some } \lambda_0 .$$

We claim

$$\text{(3.15)} \qquad \int_{|y|\ge r_0} J_{\lambda^2}(y)|q_2(x-y)|dy \to 0 \quad \text{as } \lambda \to \infty .$$

Denote the integral above by I_λ; we have

$$I_\lambda = \sum_{i=0}^{\infty} \int_{2^i r_0 \le |y| < (2^{i+1})r_0} |q_2(x-y)| J_{\lambda^2}(y)dy .$$

We may assume without loss of generality that $\lambda r_0 > R$ (where R is the constant appearing in (3.11)) and thus conclude

$$I_\lambda \le \sum_{i=0}^{\infty} \lambda^{n-2} \int_{2^i r_0 \le |y| < (2^{i+1})r_0} e^{-\delta\lambda|y|}|q_2(x-y)|dy .$$

Since $|y| \ge r_0$ there are constants $\delta' > 0$ and $K' > 0$ such that

$$I_\lambda \le \sum_{i=0}^{\infty} K'\lambda^{-1} \int_{2^i r_0 \le |y| < (2^{i+1})r_0} e^{-\delta'|y|}|q_2(x-y)|dy .$$

Further if $\lambda > \lambda'$ then δ' and K' depend only on λ', i.e. we may assume δ' and K' are fixed as $\lambda \to \infty$. We now have

$$I_\lambda \le \sum_{i=0}^{\infty} K'\lambda^{-1} \int_{2^i r_0 \le |y| < (2^{i+1})r_0} e^{-\delta' 2^i r_0}|q_2(x-y)|dy .$$

Using the estimate (3.9) of Lemma 3.2 for the ball of radius $(2^{i+1})r_0$ we conclude

$$I_\lambda \le \sum_{i=0}^{\infty} K'\lambda^{-1} K(2^{i+1}r_0)^n e^{-\delta' 2^i r_0}$$

$$\le K''\lambda^{-1} \int_0^\infty 2^{n(\xi+1)} r_0^n e^{-\delta' 2^{\xi-1} r_0} d\xi .$$

We make the change of variables, $\tau = 2^\xi$, and lump constants into one term to obtain

$$I_\lambda \le \bar{K}\lambda^{-1} \int_0^\infty \tau^{n-1} e^{-\frac{\delta' r_0}{2}\tau} d\tau .$$

It is now obvious that the integral above is bounded independent of λ and therefore there exists a constant $C > 0$ such that

$$I_\lambda \le C\lambda^{-1} .$$

This completes the proof of (3.15), which proves Lemma 3.3.

We now complete the proof of Theorem 3.1 for $q^* =$ constant. Let λ be a constant $\lambda > \lambda_0$ (λ_0 determined by Lemma 3.3). It follows from Lemma 3.3 that

$$(3.16) \qquad (-\Delta + \lambda I)^{-1}(q_2^+)(x) \le \alpha_0 + \frac{1 - \alpha_0}{2} < 1 ,$$

where $q_2^+(x) = \max\{q_2(x), 0\}$. Now consider

$$T + q^* \equiv \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} - ib_j(x)\right)^2 + (q_1(x) + q^* + q_2^-(x)) - q_2^+(x) .$$

By (3.16) and assumptions on T we may apply Theorem 2.2 to $T + q^*$ to conclude that if $u \in L^2$,

$$(T + q^* + \zeta)u = 0, \quad \text{(in the sense of distributions)}$$

and $\mathrm{Re}(\zeta) > \lambda_0$ then $u(x) \equiv 0$. Thus it immediately follows that if $u \in L^2$,

$$(T + \zeta)u = 0, \quad \text{(in the sense of distributions)}$$

and $\mathrm{Re}(\zeta) > \lambda_0 + q^*$ then $u(x) \equiv 0$. This proves $\dot{T}$ is essentially selfadjoint in the case $q^* =$ constant.

In order to complete the proof of Theorem 3.1 we shall have to prove a result analogous to Proposition 4, pg. 141 of [2]. We introduce the function space H_b^1. H_b^1 is defined as the completion of C_0^∞ in the norm

$$\|\phi\|^2_{H_b^1} \equiv \sum_{j=1}^{n} \left\|\left(\frac{\partial \phi}{\partial x_j} - ib_j\phi\right)\right\|^2_{L^2} + \|\phi\|^2_{L^2} .$$

As noted in [2] H_b^1 is a Hilbert space

$$H_b^1 \subset L^2 \subset H_b^{-1} \quad \text{and} \quad \|\phi\|_{H_b^1} \ge \|\phi\|_{H^1} .$$

It is also clear that

$$(T\phi,\phi) = \|\phi\|^2_{H_b^1} - \|\phi\|^2_{L^2} + ((q_1 - q_2)\phi,\phi) .$$

<u>Lemma 3.4</u>. Let $u \in L^2$, $Tu \in H_b^{-1}$ and $q^* =$ constant. Then $u \in H_b^1$ and there are constants C_1 and C_2 such that

$$(3.17) \qquad \|u\|_{H_b^1} \le C_1\|Tu\|_{H_b^{-1}} + C_2\{1 + (q^*)^{\frac{1}{2}}\}\|u\|_{L^2} .$$

<u>Proof</u>. For any $\phi \in C_0^\infty$ we have

$$((T + \lambda)\phi,\phi) = \|\phi\|^2_{H_b^1} + (\lambda - 1)\|\phi\|^2_{L^2} + ((q_1 - q_2)\phi,\phi) .$$

We take $\lambda = \lambda_0 + q^* + 1$ where λ_0 is the constant from Lemma 3.3. Thus

$$(3.18) \qquad ((T + \lambda)\phi,\phi) \ge \|\phi\|^2_{H_b^1} + \lambda_0\|\phi\|^2_{L^2} - (q_2\phi,\phi) .$$

By Lemma 3.3 and Theorem 1.1 we have

$$\left(1 - \frac{1 - \alpha_0}{2}\right)\|\phi\|^2_{H^1} + \lambda_0\|\phi\|^2_{L^2} - (|q_2|\phi,\phi) \ge 0 ,$$

and using this in (3.18) gives us (since $\|\cdot\|_{H_b^1} \ge \|\cdot\|_{H^1}$)

$$((T + \lambda)\phi,\phi) \geq \left(\frac{1 - \alpha_0}{2}\right)\|\phi\|^2_{H^1_b} ,$$

or

$$(T\phi,\phi) + \lambda\|\phi\|^2_{L^2} \geq \left(\frac{1 - \alpha_0}{2}\right)\|\phi\|^2_{H^1_b} .$$

If $\phi_i \to w \in H^1_b$ then $T\phi_i \to Tw \in H^{-1}_b$ and so

$$(Tw,w) + \lambda\|w\|^2_{L^2} \geq \left(\frac{1 - \alpha_0}{2}\right)\|w\|^2_{H^1_b} .$$

Consequently for any $\varepsilon > 0$ there is K_ε such that

$$K_\varepsilon\|Tw\|^2_{H^{-1}_b} + \varepsilon\|w\|^2_{H^1_b} + (\lambda_0 + q^*)\|w\|^2_{L^2} \geq \left(\frac{1 - \alpha_0}{2}\right)\|w\|^2_{H^1_b}$$

and therefore there are constants C_1 and C_2 such that

$$C_1\|Tw\|_{H^{-1}_b} + C_2\left(1 + (q^*)^{\frac{1}{2}}\right)\|w\|_{L^2} \geq \|w\|_{H^1_b} , \tag{3.19}$$

for any $w \in H^1_b$.

It only remains to show that $u \in H^1_b$. This follows for exactly the same reason as in [2]; this completes the proof of the lemma.

Proof of Theorem 3.1. Using the result for the case $q^* =$ constant and Lemma 3.4 we may now proceed exactly as in [2]. This completes the proof of the theorem.

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~~We give~~ conditions for the function $q(x)$ which guarantee uniqueness of solutions to $-\Delta u(x) + \lambda_0 u(x) - q(x)u(x) = f$, where $\lambda_0 \geq 0$ is a constant. Delta lambda sub 0 sub 0 ~~We consider separately~~ the cases for solutions in $H^{(1)}_{\sqcup}$ and $L^{(2)}_{\sqcup}$ are considered separately. In fact, ~~we~~ ~~even prove~~ existence of $H^{(1)}_{\sqcup}$ solutions under the additional assumptions $q(x) \geq 0$ and $f \in L^2$ are proven. These ~~We apply our~~ results are applied to the problem of essential → next page

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>or= is an element of L(2)

20. ABSTRACT - Cont'd.

selfadjointness for a class of Schrödinger operators and ~~we are able to prove~~ an interesting generalization of a result due to Kato is proven.

Gaveau ~~has~~ had proved a result similar to ~~ours~~ this on L^{2} uniqueness. His proof, however, ~~depends~~ depended on a probabalistic argument, while in contrast ~~we~~ there is no ~~do not use any~~ probability theory used in ~~our~~ these proofs.